Journal of Engineering Physics and Thermophysics, Vol. 74, No. 5, 2001

INVESTIGATION OF A LAMINAR BOUNDARY LAYER ON A HORIZONTAL CONTINUOUSLY MOVING PLANE SURFACE IN THE PRESENCE OF A COCURRENT FLOW

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UDC 532.526

On the basis of the stationary laminar boundary layer equations, an analysis of the external flow effect on the characteristics of the boundary layer of a continuously moving flat plate is carried out. Numerical and approximate analytical solutions of the problem have been obtained for different values of the parameter ε, which characterizes the ratio of the velocities of the moving plate and cocurrent flow. Correlation dependences have been constructed for determining the boundary-layer thickness and flow shear on the body surface.

One of the most sequentially studied problems in fluid mechanics is the simulation of flow near a flat plate placed parallel to the direction of a main stream. However, despite the major successes achieved in the solution of this problem for a laminar regime of flow [1–4], there are a number of insufficiently studied features in the structure of the flow field. For example, there are relatively few works [5–9] devoted to the mechanism of interaction of a cocurrent flow with the boundary layer of a continuously moving plane surface, i.e., the mechanism that plays a very important role in various technical and gas-dynamic problems.

Below we present the results of extensive mathematical simulation of a laminar regime of an infinite cocurrent homogeneous flow past a flat horizontal continuously moving surface in the entire range of change of ε from zero to infinity.

The initial equations are the boundary-layer equations that for a two-dimensional stationary flow of liquid have the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} . \tag{1}$$

with boundary conditions

$$y = 0 : \; v = 0 , \quad u = u_{\text{w}} ; \quad y \to \infty : \; u \to u_\infty . \tag{2}$$

We introduce the self-similar variables

$$\psi = (\nu u_\infty x)^{1/2} F(\zeta) , \quad \zeta = \left(\frac{u_\infty}{\nu x}\right)^{1/2} y . \tag{3}$$

As a result of transformations, Eqs. (1)–(2) yield

$$F''' + \frac{1}{2} F F'' = 0 , \quad F(0) = 0 , \quad F'(0) = \varepsilon , \quad F'(\infty) = 1 , \tag{4}$$

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TABLE 1. Results of Numerical Calculation of a Boundary Layer for $0 \le \varepsilon < 1$

ε	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$F''(0)$	0.332057	0.327046	0.313358	0.292432	0.265232	0.232455	0.194626	0.152159	0.105389	0.054591
$(\delta^*/x)\,\mathrm{Re}_x^{1/2}$	1.720788	1.446399	1.213270	1.009272	0.827060	0.661826	0.510233	0.369866	0.238918	0.115994
δ^*/δ^{**}	2.59110	2.21131	1.93592	1.72565	1.55912	1.42356	1.31080	1.21539	1.13351	1.06239

with the prime meaning a derivative with respect to ζ.

We note that the parameter $\varepsilon = u_w/u_\infty$ is a measure of the magnitude of the effect caused by the action of a cocurrent flow on the boundary layer: when $\varepsilon = 0$, we have a regime of flow past a stationary surface [10], and the case of $\varepsilon = \infty$ corresponds to the motion of a plate with velocity $u = u_w$ through a stagnant fluid [11]. The two-point boundary-value problem (4) was solved numerically using Heming's integration scheme, having the fourth order of accuracy, by reducing system (4) to the corresponding Cauchy problem. The lacking boundary condition $F''(0)$ was selected by the Newton–Rafson method so that the asymptotic condition $F'(\infty) = 1$ could be satisfied on the outer edge of the boundary layer to within $\sim 10^{-11}$. The data of calculations for ε changing from 0 to 0.9 are given in Table 1. Among the physical quantities of interest are the velocity distribution u and the local coefficient of friction c_f that can be found from the formulas ($\mathrm{Re}_x = u_\infty x/\nu$)

$$\frac{u}{u_\infty} = F'(\zeta)\,, \quad c_f\,\mathrm{Re}_x^{1/2} = 2F''(0)\,. \tag{5}$$

As is seen from Table 1, the coefficient of surface friction c_f, the displacement thickness δ^*, and the form-parameter of the boundary layer H decrease monotonically with increase in ε. The greatest rate of fall is typical of the quantity δ^*.

Further, we consider the behavior of the solution of Eqs. (1)–(2) for $\varepsilon > 1$. We note that for $\varepsilon = 1$ problem (4) admits integration in quadratures: $F = \zeta$ and $F' = 1$. For the regime with $\varepsilon > 1$, it is advantageous to represent the unknown functions otherwise to have the entire complex of values of ε in the range of change from 1 to infinity. This is attained by the introduction of new variables (the prime denotes the derivative with respect to η):

$$\frac{u_w - u}{u_w - u_\infty} = f'(\eta)\,, \quad v = \sqrt{(u_w - u_\infty)\,\nu}\left(\frac{1}{2}f - \frac{1}{2}f'\eta\right)x^{-1/2}\,, \quad \eta = \left(\frac{u_w - u_\infty}{\nu}\right)^{1/2} x^{-1/2}\,y\,. \tag{6}$$

In accordance with the transformations (6), Eqs. (1)–(2) can be rewritten as

$$f''' + \frac{1}{2}f''\left(\frac{\varepsilon}{\varepsilon - 1}\eta - f\right) = 0\,, \quad f(0) = 0\,, \quad f'(0) = 0\,, \quad f'(\infty) = 1\,. \tag{7}$$

The boundary-value two-point problem (7) can be solved by the method of successive approximations. The idea of the approach rests on the division of $f(\eta)$ into n functions: $f = f_0 + f_1 + f_2 + \ldots$, where, as the zero approximation, we take the solution corresponding to the situation where the boundary conditions have the form $f_0(0) = 0$, $f_0'(0) = 1$, $f_0'(\infty) = 1$. As a result we obtain an infinite sequence of problems:

• zero approximation

$$f_0''' + \frac{1}{2}f_0''\left(\frac{\varepsilon}{\varepsilon - 1}\eta - f_0\right) = 0\,, \quad f_0(0) = 0\,, \quad f_0'(0) = 1\,, \quad f_0'(\infty) = 1\,;$$

TABLE 2. Results of a Numerical Calculation of a Boundary Layer for $1 < \varepsilon \le \infty$

ε	$f''(0)$	η_u	ε	$f''(0)$	η_u
1.25	1.214624	1.7377	20	0.462985	5.8256
2	0.720585	3.0993	30	0.456468	5.9926
3	0.599331	3.9029	50	0.451332	6.1352
4	0.552800	4.3614	100	0.447524	6.2478
5	0.527900	4.6664	200	0.445633	6.3014
6	0.512328	4.8869	500	0.444501	6.3426
8	0.493865	5.1870	1000	0.444125	6.3546
10	0.483263	5.3832	∞	0.443748	6.3674

• first approximation

$$f_1''' + \frac{1}{2} f_1'' \left(\frac{\varepsilon}{\varepsilon - 1} \eta - f_0 \right) - \frac{1}{2} f_0'' f_1 = 0 \, , \quad f_1 (0) = 0 \, , \quad f_1' (0) = -1 \, , \quad f_1' (\infty) = 0 \, ; \tag{8}$$

• second approximation

$$f_2''' + \frac{1}{2} f_2'' \left(\frac{\varepsilon}{\varepsilon - 1} \eta - f_0 \right) - \frac{1}{2} f_0'' f_2 = \frac{1}{2} f_1 f_1'' \, , \quad f_2 (0) = 0 \, , \quad f_2' (0) = 0 \, , \quad f_2' (\infty) = 0 \, ;$$

• ith approximation ($i = \overline{3, n}$)

$$f_i''' + \frac{1}{2} f_i'' \left(\frac{\varepsilon}{\varepsilon - 1} \eta - f_0 \right) - \frac{1}{2} f_0'' f_i = \frac{1}{2} \sum_{j=1}^{i-1} f_j f_{i-j}'' \, , \quad f_i (0) = 0 \, , \quad f_i' (0) = 0 \, , \quad f_i' (\infty) = 0 \, .$$

It can be easily seen that $f_0 = \eta$ and $f_0' = 1$. This solution describes a homogeneous flow which is parallel to the plate surface. Knowing f_0, by direct integration of (8) we find the functions f_1 and f_1':

$$f_1 = \eta \left(\mathrm{erf} \left(\frac{\eta}{2 \sqrt{\varepsilon - 1}} \right) - 1 \right) + \sqrt{\frac{\varepsilon - 1}{\pi}} \left(2 \exp \left(- \frac{\eta^2}{4 (\varepsilon - 1)} \right) - 2 \right), \quad f_1' = \mathrm{erf} \left(\frac{\eta}{2 \sqrt{\varepsilon - 1}} \right) - 1 \, . \tag{9}$$

And, finally, taking into account relation (9), after simple but cumbersome calculations we write out an expression for f_2' in the final form:

$$f_2' = \frac{\varepsilon - 1}{\pi} \left[\frac{2 - \pi}{2} \mathrm{erf} \left(\frac{\eta}{2 \sqrt{\varepsilon - 1}} \right) + 2 \exp \left(- \frac{\eta^2}{4 (\varepsilon - 1)} \right) - 1 - \exp \left(- \frac{\eta^2}{2 (\varepsilon - 1)} \right) \right] +$$
$$+ \sqrt{\frac{\varepsilon - 1}{\pi}} \left[\int_0^{\eta} \exp \left(- \frac{\eta^2}{4 (\varepsilon - 1)} \right) \mathrm{erf} \left(\frac{\eta}{2 \sqrt{\varepsilon - 1}} \right) d\eta \, + \right.$$
$$\left. + \frac{1}{2} \eta \exp \left(- \frac{\eta^2}{4 (\varepsilon - 1)} \right) - \frac{1}{2} \eta \exp \left(- \frac{\eta^2}{4 (\varepsilon - 1)} \right) \mathrm{erf} \left(\frac{\eta}{2 \sqrt{\varepsilon - 1}} \right) \right] . \tag{10}$$

Thus, we have found the first three terms of the series that make it possible to construct the velocity profiles $(u_{\mathrm{w}} - u)/(u_{\mathrm{w}} - u_{\infty})$ for different values of ε and also to compute the flow shear on the plate

$\frac{\tau_w}{\rho u_\infty^2}\mathrm{Re}_x^{1/2} = -(\varepsilon-1)^{3/2} f''(0)$ (the "minus" sign means that the plane surface has propulsion) and the boundary-layer thickness $\delta_u = \eta_u \left(\frac{\nu x}{u_w - u_\infty}\right)^{1/2}$, which is defined as the distance from the wall to the point at which $f' = 0.99$. By virtue of Eqs. (9)–(10) we have

$$f''(0) = \frac{1}{\sqrt{\pi(\varepsilon-1)}}\left(1+\frac{1}{\pi}(\varepsilon-1)+...\right), \quad \eta_u = 3.6427\sqrt{\varepsilon-1}\,[1-0.2005\,(\varepsilon-1)+...]\,. \tag{11}$$

Calculations from the above formulas have shown that an increase in ε entails an increase in η_u and decrease in $f''(0)$. Naturally, the working formulas approximate the exact solution only in the range $1<\varepsilon<\varepsilon^*$. Moreover, Eq. (11) can be used only for determining the parametric region of the application of an approximate solution. Employing the requirement that the second term in Eq. (11) must never exceed unity, we find the upper limit for ε: $\varepsilon^* \approx 4$.

However, this shortcoming can be obviated by resorting to the procedure of restoring the main properties of the solution on the basis of several first terms of the series [12]. This procedure corresponds to the summing up of the infinite numerical sequence contained in the main series:

$$f''(0) = \frac{1}{\sqrt{\pi\,(\varepsilon-1)}}\left(1+\frac{155}{121}(\varepsilon-1)+\frac{199}{520}(\varepsilon-1)^2\right)^{1/4}, \quad \eta_u = \frac{3.6427\sqrt{\varepsilon-1}}{(1+0.8068\,(\varepsilon-1)+0.1071\,(\varepsilon-1)^2)^{1/4}}\,. \tag{12}$$

It is seen from relations (12) that the calculation of the boundary layer in terms of the variables of Eq. (6) has the following distinctive feature: when $\varepsilon\to\infty$ (and, consequently, $u_\infty\to 0$), the quantity $f''(0)(\eta_u)$ tends not to zero (infinity), but rather to a definite limit:

$$\varepsilon\to\infty:\ f''(0) = \left(\frac{199}{520\pi^2}\right)^{1/4}, \quad \eta_u = 6.3676\,.$$

This is due to the fact that Eqs. (7) include the regime $\varepsilon = \infty$ as a particular case. It is of interest to note that when $\varepsilon \ge 30$, the parameter ε exerts an insignificant effect on the characteristics sought (at most 3% for $f''(0)$ and 6% for η_u). Physically, this means that when $\varepsilon > 30$, the structure of the boundary layer differs insignificantly from the structure of the boundary layer with $\varepsilon = \infty$ and, therefore, the effect of the cocurrent stream on the flow in the region with $30<\varepsilon\le\infty$ can be neglected. Consequently, the results obtained for the boundary layer on a plane surface continuously moving with a constant velocity in a quescent fluid [13] can be used for the analysis of a flow in the presence of a cocurrent stream ($\varepsilon > 30$).

We also carried out a numerical integration of problem (7) to find out to what extent formulas (12) describe actual distributions of the characteristics sought. It follows from comparison that the relations calculated from analytical equations (12) and with the use of the numerical procedure (Table 2) practically coincide: the difference in the values does not exceed 0.04% for $f''(0)$ and 0.09% for η_u. If we represent the numerical data of the regimes with $\varepsilon > 1$ in the form of the ratio $|\tau_w|/\tau_{w0}$, where τ_{w0} is the friction stress on the plate for $\varepsilon = 0$, then the graph of the function $|\tau_w|/\tau_{w0}$ has a monotonic character: it starts from zero, attains unity for $\varepsilon = 1.51150$, and then increases with ε. Therefore, with other conditions being equal, the shear stresses on the wall for the flow with $u_w > 1.51150u_\infty$ are higher than for the boundary layer formed on a stationary plane immersed in a flow.

An approximate solution of problem (4) can be constructed likewise. Here the formulas that determine the dependences of the quantities $F''(0)$ and $\frac{\delta^*}{x}\mathrm{Re}_x^{1/2}$ on ε have the form

$$F''(0) = \frac{1}{2}\left(\frac{11}{18\pi}\right)^{1/4}(1-\varepsilon)\left(1+\frac{109}{27}\varepsilon+\frac{89}{27}\varepsilon^2\right)^{1/4}, \quad \frac{\delta^*}{x}\mathrm{Re}_x^{1/2} = 5\left(\frac{9}{65\pi^2}\right)^{1/4}\frac{(1-\varepsilon)}{\left(1+\frac{1255}{416}\varepsilon+\frac{579}{416}\varepsilon^2\right)^{1/4}}. \quad (13)$$

The calculations have shown that within the range of ε from zero to unity Eqs. (13) yield an error not exceeding 0.04%. Thereafter, assuming that $\varepsilon = 0$, from Eqs. (13) we obtain

$$F''(0) = \frac{1}{2}\left(\frac{11}{18\pi}\right)^{1/4}, \quad \frac{\delta^*}{x}\mathrm{Re}_x^{1/2} = 5\left(\frac{9}{65\pi^2}\right)^{1/4}.$$

These results practically coincide with numerical values: the absolute error is equal to about $4\cdot10^{-7}$. In conclusion, we find the pulse-loss thickness δ^{**}. Since

$$\frac{\delta^{**}}{x}\mathrm{Re}_x^{1/2} = \int_0^\infty (F' - F'^2)\, d\zeta\,,$$

integration of Eq. (4) from 0 to ∞ yields

$$\int_0^\infty (F' - F'^2)\, d\zeta = 2F''(0) - F(0)\,(1 - F'(0))\,.$$

Consequently, in the case of an impenetrable plane surface the value of the dimensionless pulse-loss thickness is equal to $2F''(0)$.

Thus, in the boundary-layer approximation the problem of the laminar mode of nongradient cocurrent homogeneous flow past a continuously moving (with constant velocity) infinitely thin flat horizontal plate has been solved. The entire range of change of the parameter $\varepsilon = u_w/u_\infty$, i.e., from zero ($u_w = 0$) to infinity ($u_\infty = 0$), has been investigated. Numerical data on the coefficient of skin friction, displacement thickness, form-parameter, and boundary-layer thickness in the ranges $0 \le \varepsilon < 1$ and $1 < \varepsilon \le \infty$ are given. An approximate analytical solution for the velocity field has been constructed that made it possible to write out in an explicit form correlations to calculate various characteristics of the boundary layer. It has been established that shear stresses on the wall for $\varepsilon > 1.5115$ are higher than the values for the case of flow past a motionless surface.

NOTATION

u and v, longitudinal and transverse velocity components; x and y, longitudinal and transverse coordinates; u_w and u_∞, velocity of the plate and of the main stream; ψ, stream function; ν, kinematic viscosity coefficient; Re_x, Reynolds number; τ_w, flow shear on the wall; ρ, density; $c_f = \tau_w/\frac{\rho u_\infty^2}{2}$, local coefficient of friction; δ_u, boundary-layer thickness; $\delta^* = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right)dy$, displacement thickness; $\mathrm{erf}\,(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt$, integral of probabilities; $\delta^{**} = \int_0^\infty \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy$, pulse-loss thickness; $H = \delta^{*}/\delta^{**}$, form-parameter of the boundary layer.

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